Week 8

Recall: Taylor polynomial of f at a

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$f(x) = p_n(x) + R_n(x)$$

Taylor's theorem:

If $x > a$, $f^{(n)}$ exists and is continuous on $[a,x]$

(similar statement for $x < a$) $f^{(n+1)}$ exists on (a,x)

Then $\exists\, c \in (a,x)$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

i.e. $$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

eg Approximate $\cos(0.1)$ using Taylor polynomials of $\cos x$ at 0. Find an n such that the error $< 10^{-6}$

Sol Let $f(x) = \cos x$

$p_n(x)$ = Taylor polynomial of order n of f at 0

$$f(x) = p_n(x) + R_n(x)$$

Also, f is infinitely differentiable on $\mathbb{R}$

Taylor's theorem

$\Rightarrow \exists\, c \in (0, 0.1)$ such that

$$R_n(0.1) = \frac{f^{(n+1)}(c)}{(n+1)!}(0.1-0)^{n+1}$$

$$= \frac{f^{(n+1)}(c)}{(10)^{n+1}(n+1)!}$$

Note $f^{(n+1)}(c) = \pm \sin c$ or $\pm \cos c$

$\Rightarrow |f^{(n+1)}(c)| \le 1$

$\Rightarrow$ error $= |R_n(x)| \le \dfrac{1}{10^{n+1}(n+1)!} < \dfrac{1}{10^{n+1}}$

If $n \ge 5$, then error $< \dfrac{1}{10^6} = 10^{-6}$

Rmk $p_5(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

$p_5(0.1) = 0.9950041666\ldots$

$\cos(0.1) = 0.99500416527\ldots$

We can look at error at other point similarly

Fix $x \neq 0$.

Taylor Thm $\Rightarrow R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$

for some c between 0 and x

$|f^{(n+1)}(c)| \le 1$

$\Rightarrow |R_n(x)| \le \dfrac{|x|^{n+1}}{(n+1)!}$

$\dfrac{|x|}{n} \to 0$ as $n \to \infty$

$\dfrac{|x|}{1} \cdot \dfrac{|x|}{2} \cdots \dfrac{|x|}{3} \cdots \dfrac{|x|}{n} \cdot \dfrac{|x|}{n+1}$

Observations:

1. $\lim_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$

Sandwich Thm $\Rightarrow R_n(x) \to 0$ as $n \to \infty$

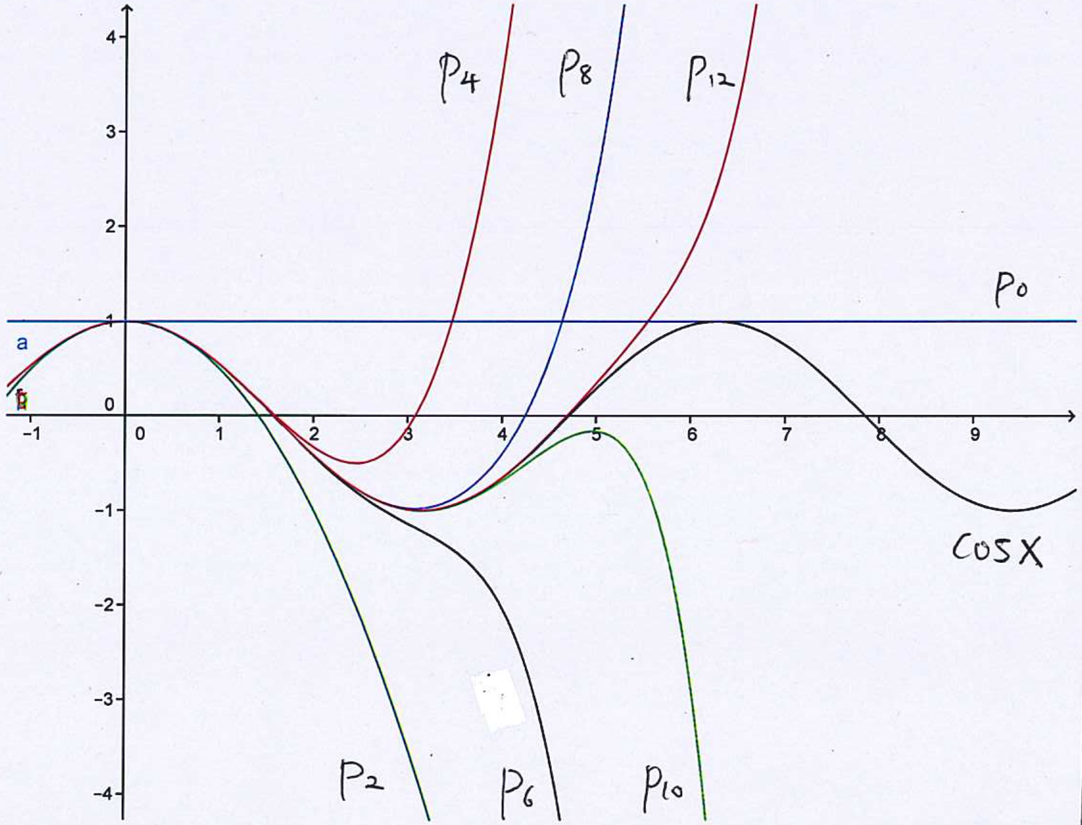
$\Rightarrow p_n(x) \to f(x)$ as $n \to \infty$

2. $\dfrac{|x|^{n+1}}{(n+1)!} \to 0$ more slowly for large $|x|$

suggests

$p_n(x) \to f(x)$ more slowly for larger $|x|$

See graph on next page



Taylor polynomials of $\cos x$ at 0

Note: For each x, $p_n(x) \to \cos x$

The convergence is slower for larger $|x|$

eg $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

① Find Taylor polynomial of f at 0

② What does Taylor's theorem say about the approximation $f(x_0) \approx p_n(x_0)$ for

a. $x_0 = 0.1$

b. $x_0 = -2$

c. $x_0 = 2$

Sol

①

$f'(x) = \frac{1}{(1-x)^2} \quad f''(x) = \frac{2}{(1-x)^3}$

$\vdots$

$f^{(k)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow f^{(k)}(0) = k!$

$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} x^k$

$= 1 + x + x^2 + \cdots + x^n$

2a Taylor's thm

$$\Rightarrow R_n(0.1) = \frac{f^{(n+1)}(c)}{(n+1)!}(0.1-0)^{n+1} \text{ for some } 0<c<1$$

$$= \frac{\frac{(n+1)!}{(1-c)^{n+1}}}{(n+1)!}(0.1)^{n+1}$$

$$= \frac{(0.1)^{n+1}}{(1-c)^{n+2}}$$

$$< \frac{(0.1)^{n+1}}{(0.9)^{n+2}} = \frac{1}{0.9}\left(\frac{1}{9}\right)^{n+1}$$

Sandwich thm $(0 \le R_n(0.1))$

$\Rightarrow R_n(0.1) \to 0$ as $n \to \infty$

$\Rightarrow p_n(0.1) \to f(0.1)$ (Good Appoxmation!)

Rmk $f(0.1) = \frac{1}{1-0.1} = 1.1111\cdots\cdots$

$$p_n(0.1) = 1 + 0.1 + 0.1^2 + \cdots + 0.1^n = 1.\underbrace{111\cdots1}_{n\ 1\text{'s}}$$

2b Similarly, by Taylor's thm

$$R_n(-2) = \frac{(-2)^{n+1}}{(1-c)^{n+2}} \text{ for some } -2<c<0$$

$$\Rightarrow |R_n(-2)| = \frac{2^{n+1}}{(1-c)^{n+2}}$$

We don't know what exactly c is

Large if c is close to 0

$\therefore$ We do not know from Taylor's theorem whether $\lim_{n\to\infty} p_n(-2) = f(-2)$

2c $a=0,\ x_0=2$

To apply Taylor's thm, we need $f^{(n)}(x)$ exists on $(0,2)$. However, f is not even defined at 1

$\Rightarrow$ Taylor's theorem cannot be applied.

Rmk for 2b, 2c

$$p_n(-2) = 1 - 2 + 2^2 - 2^3 + 2^4 - \cdots + (-2)^n$$

$$p_n(2) = 1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n$$

Divergent as $n\to\infty$ in both cases

Taylor Series

Suppose f is infinitely differentiable at a

We saw examples that $\lim_{n\to\infty} p_n(x) = f(x)$,

where $p_n(x)$ = n-th order Taylor poly of f at a

Define the Taylor series of f at a to be the expression

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{p_n(x)}$$

Rmk ① An expression of the form

$\sum_{k=0}^{\infty} a_k x^k$ is called a power series

② If $R_n(x) \to 0$ on an interval I, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \text{ on } I$$

Warning: $\exists$ function f such that $\lim_{n\to\infty} R_n(x) \neq 0$ for any $x \neq a$

Examples of functions which equal to their Taylor series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots \quad \text{for } |x|<1$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots \quad \text{for } |x|<1$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for } x \in \mathbb{R}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for } x \in \mathbb{R}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{for } x \in \mathbb{R}$$

$$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \quad \text{for } |x-1|<1$$

Equivalently, replacing $x-1$ by x,

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x|<1$$

Except for $\ln x$, the Taylor series above are centred at 0

Operations on Taylor series

Addition/Subtraction

(Recall: $(f+g)' = f' + g'$)

$\sin x + \cos x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\right)$$

$$+ \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots\cdots$$

$\frac{1}{1-x} - \frac{1}{1+x}$ $\quad (|x|<1)$

$$= (1+x+x^2+x^3+\cdots) - (1-x+x^2-x^3+\cdots)$$

$$= 2 + 2x^2 + 2x^4 + \cdots\cdots$$

$$= 2(1+x^2+x^4+\cdots)$$

$$= 2 \cdot \frac{1}{1-x^2}$$

(Note $\frac{1}{1-x} - \frac{1}{1+x} = \frac{2}{1-x^2}$)

Multiplication

(Recall: Leibniz rule $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$)

$(e^x)(e^x)$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$+ x + x^2 + \frac{x^3}{2} + \cdots$$

$$+ \frac{x^2}{2} + \frac{x^3}{2} + \cdots$$

$$+ \frac{x^3}{6} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots\cdots$$

Ex

Try similar computation for

$\sin x \cos x = \frac{1}{2}\sin 2x$

Note

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

equal $(\because (e^x)^2 = e^{2x})$

$$= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots$$

Division

eg: $\sec x = \dfrac{1}{\cos x}$

Method I

Let $\sec x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

Then

$$1 = (\cos x)(\sec x)$$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$- \frac{a_0}{2}x^2 - \frac{a_1}{2}x^3 - \frac{a_2}{2}x^4 - \cdots$$

$$+ \frac{a_0}{24}x^4 + \cdots$$

Comparing coefficients

$a_0 = 1$

$a_1 = 0$

$a_2 - \frac{a_0}{2} = 0 \Rightarrow a_2 = \frac{1}{2}$

$a_3 - \frac{a_1}{2} = 0 \Rightarrow a_3 = 0$

$a_4 - \frac{a_2}{2} + \frac{a_0}{24} = 0 \Rightarrow a_4 = \frac{5}{24}$

$$\Rightarrow \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$$

Method II

(*) $\dfrac{1}{1-y} = 1 + y + y^2 + \cdots$

$$\sec x$$

$$= \frac{1}{\cos x}$$

$$= \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right)}$$

show only terms of deg ≤ 4

$$\overset{(*)}{=} 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + \left(\frac{x^2}{2} - \frac{x^2}{24} + \cdots\right)^2 + \cdots$$

$$= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + \left(\frac{x^4}{4} + \cdots\right) + \cdots$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$$

same

Composition

calculate up to deg 3

$$e^{\ln(1+x)}$$

$$= 1 + \ln(1+x) + \frac{[\ln(1+x)]^2}{2} + \frac{[\ln(1+x)]^3}{6} + \cdots$$

$$= 1 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) + \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)^2$$

$$+ \frac{1}{6}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)^3$$

$$= 1 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) + \frac{1}{2}\left(x^2 - x^3 + \cdots\right)$$

$$+ \frac{1}{6}\left(x^3 + \cdots\right)$$

$$= 1 + x + 0 \cdot x^2 + 0x^3 + \underbrace{\cdots}$$

all higher terms are zero

$$\therefore e^{\ln(1+x)} = 1 + x$$

Differentiation / Integration

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$(\sin x)' = 1 - 3 \cdot \frac{x^2}{3!} + 5 \cdot \frac{x^4}{5!} - 7 \cdot \frac{x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \cos x$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\int \frac{1}{1+x}\,dx = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + C$$

$$= \ln(1+x) + C$$

Taking limit

eg $\lim_{x\to 0} \dfrac{e^x-1-\sin x}{\cos x-1}$

$$e^x-1-\sin x = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)-1-\left(x-\frac{x^3}{3}+\cdots\right)$$

lowest order term $\rightarrow$ $= \dfrac{x^2}{2} + HOT_3$ ← stands for higher order terms of deg ≥ 3

$$\cos x - 1 = \left(1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots\right)-1$$

$$= -\frac{x^2}{2} + HOT_4$$

$$\frac{e^x-1-\sin x}{\cos x-1} = \frac{\frac{x^2}{2}+HOT_3}{-\frac{x^2}{2}+HOT_4}$$

$$= \frac{\frac{1}{2}+HOT_1}{-\frac{1}{2}+HOT_2} \rightarrow \frac{\frac{1}{2}+0}{-\frac{1}{2}+0} = -1$$

as $x\to 0$

$$\therefore \lim_{x\to 0}\frac{e^x-1-\sin x}{\cos x-1} = -1$$

eg Find the third order Taylor polynomial of $\dfrac{\sin x}{x}$ at π by consider Taylor series.

Sol $\dfrac{\sin x}{x} = \dfrac{-\sin(x-\pi)}{\pi+(x-\pi)} = -\dfrac{1}{\pi}\cdot\dfrac{\sin(x-\pi)}{1+\frac{x-\pi}{\pi}}$

$$\therefore \frac{\sin x}{x} = -\frac{1}{\pi}\underbrace{\left((x-\pi)-\frac{(x-\pi)^3}{6}+\cdots\right)}_{\sin(x-\pi)}\underbrace{\left(1-\frac{x-\pi}{\pi}+\left(\frac{x-\pi}{\pi}\right)^2-\left(\frac{x-\pi}{\pi}\right)^3+\cdots\right)}_{\frac{1}{1+\frac{x-\pi}{\pi}}}$$

$$= -\frac{1}{\pi}\left[(x-\pi)-\frac{(x-\pi)^2}{\pi}+\frac{(x-\pi)^3}{\pi^2}-\frac{(x-\pi)^3}{6}+HOT_4\right]$$

$$= -\frac{1}{\pi}(x-\pi)+\frac{1}{\pi^2}(x-\pi)^2+\frac{1}{\pi}\left(\frac{1}{6}-\frac{1}{\pi^2}\right)(x-\pi)^3 + HOT_4$$

$\therefore p_3(x)$

$$= -\frac{1}{\pi}(x-\pi)+\frac{1}{\pi^2}(x-\pi)^2+\frac{1}{\pi}\left(\frac{1}{6}-\frac{1}{\pi^2}\right)(x-\pi)^3$$

Binomial Series

A binomial series is the Taylor series of

$f(x) = (1+x)^\alpha$ at 0 $\quad \alpha \in \mathbb{R}$

Note $f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)(1+x)^{\alpha-k}$

$f^{(k)}(0) = \alpha(\alpha-1)\cdots(\alpha-k+1)$

Binomial Series:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{for } |x|<1$$

where $\displaystyle\binom{\alpha}{k} = \frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$

Rmk If $\alpha = n$ is a positive integer

$$\binom{n}{k} = \begin{cases} \dfrac{n(n-1)\cdots(n-k+1)}{k!} = \dfrac{n!}{k!(n-k)!} & \text{if } k \le n \\[2ex] \dfrac{n(n-1)\cdots\overset{0}{(n-n)}\cdots(n-k+1)}{k!} = 0 & \text{if } k > n \end{cases}$$

Binomial Series reduces to Binomial expansion

eg $\displaystyle\binom{-1}{k} = \frac{(-1)(-2)\cdots(-k)}{(1)(2)\cdots(k)} = (-1)^k$

$\displaystyle\Rightarrow \frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$\sqrt{1+x} = (1+x)^{\frac{1}{2}}$

$\displaystyle = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$

Q Write down a general formula for this